An inverse source problem for the convection-diffusion equation

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Abstract

Purpose – To develop a numerical technique for solving the inverse source problem associated with the constant coefficients convection-diffusion equation.

Design/methodology/approach – The proposed numerical technique is based on the boundary element method (BEM) combined with an iterative sequential quadratic programming (SQP) procedure. The governing convection-diffusion equation is transformed into a Helmholtz equation and the ill-conditioned system of equations that arises after the application of the BEM is solved using an iterative technique.

Findings – The iterative BEM presented in this paper is well-suited for solving inverse source problems for convection-diffusion equations with constant coefficients. Accurate and stable numerical solutions were obtained for cases when the number of sources is correctly estimated, overestimated, or underestimated, and with both exact and noisy input data.

Research limitations/implications – The proposed numerical method is limited to cases when the Péclet number is smaller than 100. Future approaches should include the application of the BEM directly to the convection-diffusion equation.

Practical implications – Applications of the results presented in this paper can be of value in practical applications in both heat and fluid flow as they show that locations and strengths for an unknown number of point sources can be accurately found by using boundary measurements only.

Originality/value – The BEM has not as yet been employed for solving inverse source problems related with the convection-diffusion equation. This study is intended to approach this problem by combining the BEM formulation with an iterative technique based on the SQP method. In this way, the many advantages of the BEM can be applied to inverse source convection-diffusion problems.

Keywords Boundary-elements methods, Convection, Numerical analysis

Paper type Research paper

Introduction

Environmental concerns have been the focus of both public and scientific interest for about four decades, and this interest appears to be increasing. Among these concerns, water pollution plays a very important role, as water is one of our most important

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natural resources and there are many conflicting demands upon it. All over the world, rivers and lakes are contaminated by several types of pollution, resulting in high health risk to people, animals and plants that are exposed to these waters. Knowing the origin of the source of contamination is probably the most important aspect when attempting to understand, and therefore to control, the pollution transport process. Thus, a challenging issue in environmental problems is the identification of sources of pollution in waters. The aim of this paper is to assist in the development of the necessary techniques to solve this practical problem, namely the source identification problem, when having only a limited amount of measurement data taken from the water. This problem assumes the knowledge of some boundary values for the concentration and/or its normal derivative on each part of the boundary and also the existence of some point sources of pollution of unknown location and strength. With these premises, the location and the strengths of the sources are required to be found. It is important to note that the method, as presented in this study, can also be applied to identify unknown heat sources using some temperature and heat flux boundary values on the boundary of the domain as input data.

Mathematically, the source identification problem is an inverse problem. The shortest definition of inverse problems describes them as discovering the cause from a known result. Hence, in fact, all problems of observed data interpretation are inverse. Inverse problems are concerned with the determination of inputs or sources from observed/measured outputs or responses, in contrast to direct problems, where the situation is vice versa.

In this study, the governing equation for the pollution process is taken to be the steady-state convection-diffusion equation. Different numerical methods for solving this equation have been used in order to approach both contaminant and heat source identification problems. In Gorelick *et al.* (1983), an optimization approach for identifying sources of groundwater pollution, based on least squares regression and linear programming for the least absolute error estimation, is employed. Other methods that may be employed are as follows: probabilistic approaches using random walk particle methods (Bagtzoglou *et al.*, 1992), stochastic differential equations (Wilson and Liu, 1994), or Bayesian theory combined with geostatistical techniques (Snodgrass and Kitanidis, 1997), Tikhonov regularisation (Skaggs and Kabala, 1994), quasi-reversibility method (Skaggs and Kabala, 1995), minimum relative entropy (Woodbury and Ulrych, 1996), a Fourier-based inverse technique (Birchwood, 1999), non-linear least squares method (Alapati and Kabala, 2000), etc. Good literature reviews on contaminant source identification methods can be found in Atmadja and Bagtzoglou (2001) and Michalak and Kitanidis (2003).

The boundary element method (BEM) has been successfully employed in order to solve direct convective heat transfer problems and inverse heat source diffusion problems. In DeFigueiredo and Wrobel (1990) and Gupta *et al.* (1994), the BEM formulations for the steady-state convection-diffusion equation with constant and variable velocities, respectively, have been investigated when applied to direct convective heat transfer problems, while in LeNiliot and Lefèvre (2001, 2004), the BEM for the heat diffusion equation has been combined with an iterative algorithm built to minimize a cost function in order to solve inverse problems for identifying point heat sources. The BEM has many advantages when compared to other numerical methods. Probably one of the most important is the fact that the discretisations are restricted only to the boundary, thus reducing the quantity of data necessary to solve a problem. However, the BEM has not as yet been employed in order to solve inverse source problems related with the convection-diffusion equation. This study is intended to approach this very important and interesting problem, with many practical applications in both heat and fluid flow, by combining the BEM formulation with an iterative technique based on the sequential quadratic programming (SQP) method.

Mathematical formulation

The practical problem we wish to model in this paper is the case of water pollution caused by some point sources. A point source pollutant, as opposed to a non-point or dispersed source pollutant, is one that enters the water from a small pipe, channel, or some other confined and localised source. Here, small means that the diameter of the pipe is small compared to the width of the river. The most common example of a point source of pollutants is a small pipe that discharges sewage into a stream or river. Although point source pollutants are easier to deal with than non-point source pollutants, there are many situations when these point sources cannot be localised for a variety of reasons. Our investigations will focus on these cases, when it is assumed that the pollution has occurred as a result of some point sources whose strengths and locations are unknown.

Let us consider a bounded domain $\Omega \subset \mathbb{R}^d$ and we assume that its boundary Γ consist of two parts, S_1 and S_2, such that $\Gamma = S_1 \cup S_2$, where $S_1, S_2 \neq \emptyset$ and $S_1 \cap S_2 \neq \emptyset$.

The following steady-state convection-diffusion equation with constant coefficients is considered:

$$\sum_{m=1}^{d} \frac{\partial^2 c}{\partial x_m^2}(\underline{x}) - \sum_{m=1}^{d} u_m \frac{\partial c}{\partial x_m}(\underline{x}) - kc(\underline{x}) + \sum_{l=1}^{N_s} \phi_l \delta(\underline{x} - \underline{x}_l) = 0, \quad \underline{x} \in \Omega \tag{1}$$

where $c(\underline{x})$ is the concentration of the pollutant, u_m is the x_m component of the fluid velocity, k is a decay parameter, N_s is the number of sources, δ is the Dirac delta function and ϕ_l and $\underline{x}_l$ are the lth source strength and location, respectively. In this study u_m and k are assumed to be given constants, while the constants ϕ_l and the vectors $\underline{x}_l$ are to be found based on some given boundary conditions associated with equation (1). It should be mentioned that if u_m and k are variable functions then the standard BEM cannot be employed. The reason for this is that a fundamental solution for the convection-diffusion operator with variable coefficients is not known. In these cases other numerical methods could be employed, e.g. the dual reciprocity method. However, as this paper aims to develop the BEM for the solution of inverse source convection-diffusion problems, the present study is restricted to the constant-coefficient case.

Based on the change of variable:

$$c = v \exp\left(\frac{\underline{u} \cdot \underline{x}}{2}\right), \tag{2}$$

where the vector $\underline{u}$ is defined as $\underline{u} = (u_1, u_2, \ldots, u_d)$, equation (1) may be recast as follows:

$$\sum_{m=1}^{d} \frac{\partial^2 v}{\partial x_m^2}(\underline{x}) + \mu^2 v(\underline{x}) = -\exp\left(-\frac{\underline{u}\cdot\underline{x}}{2}\right)\sum_{l=1}^{N_s} \phi_l \delta(\underline{x} - \underline{x}_l), \tag{3}$$

where $\underline{x} \in \Omega$ and $\mu = i\beta$ is a purely imaginary number with:

$$\beta = \left(k + \frac{\sum_{m=1}^{d} u_m^2}{4} \right)^{1/2}. \tag{4}$$

We assume that associated with equation (3) some boundary conditions are specified such that information is available on each part of the boundary. This information is used in order to find the strengths ϕ_l and the location $\underline{x}_l$ of the sources of pollution. In practice, this means that on each part of the boundary of the domain under investigation some information is known, obtained by either taking measurements or by imposing physical considerations, e.g. the flux is zero on the banks of a river, while the point sources of the pollution are unknown and therefore sought. There are numerous different techniques for measurement and analysis of water contaminants, i.e. physical, chemical, electrochemical, or bioanalytical methods, chromatography, etc. for more details, see, for example, Greyson (1990). Mathematically, the boundary conditions can be written as follows:

$$v(\underline{x}) = \tilde{v}(\underline{x}), \quad \underline{x} \in S_1, \tag{5}$$

$$q = \frac{\partial v}{\partial n}(\underline{x}) = \tilde{q}(\underline{x}), \quad \underline{x} \in S_2, \tag{6}$$

where $\tilde{v}$ and $\tilde{q}$ are prescribed functions of $\underline{x}$. It can be seen that on some part of the boundary both v and $\partial v/\partial n$ are specified, as $S_1 \cap S_2 \neq \emptyset$.

The iterative BEM

The standard BEM procedure using constant boundary elements (CBEs) (Brebbia *et al.* 1984) is applied to equation (3) and, using the characteristics of the Dirac delta function, the following boundary integral equation is obtained for each node i, $i = \overline{1, N}$:

$$\eta_i v_i + \int_{\Gamma} v E' \, d\Gamma - \int_{\Gamma} q E \, d\Gamma = -\sum_{l=1}^{N_s} \exp\left(-\frac{\underline{u}\cdot\underline{x}_l}{2}\right)\phi_l E(\underline{x}, \underline{x}_l) \tag{7}$$

where $\eta_i = \alpha_i/2\pi$, with α_i the angle between the left and the right tangents on Γ at the boundary node i for $i = \overline{1, N}$. In particular, if Γ is smooth then $\alpha_i = \pi$ for $i = \overline{1, N}$. The function $E(\underline{x}, \underline{y})$ is the fundamental solution for the Helmholtz operator $\nabla^2 + \mu^2$ and is given by:

$$E(\underline{x}, \underline{y}) = \frac{i}{4} H_0^{(1)}(\mu r(\underline{x}, \underline{y})), \tag{8}$$

where:

$$r(\underline{x},\underline{y}) = |\underline{x} - \underline{y}| = \left[\sum_{m=1}^{d} (x_m - y_m)^2\right]^{1/2}$$

is the geodesic distance and $H_0^{(1)}$ is the zero-order Hankel function of the first kind, see, for example, Abramowitz and Stegun (1972).

After integrating over each boundary element, equation (7) can be written in terms of the nodal values as follows:

$$\eta_i v_i + \sum_{k=1}^{N} H_{ik} v_k - \sum_{k=1}^{N} G_{ik} q_k = -\sum_{l=1}^{N_s} I_{il}\phi_l, \quad i = \overline{1,N}, \tag{9}$$

where H_{ik} and G_{ik} are the usual resultants of integration over the boundary elements (Brebbia *et al.* 1984), and I_{il} is specific to the inverse source problem for the steady-state convection-diffusion equation. Compared to the previous coefficients, the coefficient I_{il} is not the result of an integral, its expression being the following:

$$I_{il} = \exp\left(-\frac{\underline{u}\cdot\underline{x}_l}{2}\right)E(\underline{x}_i,\underline{x}_l), \quad i = \overline{1,N}, \ l = \overline{1,N_s}. \tag{10}$$

After application to all the boundary nodes $i = \overline{1,N}$, and incorporating the terms η_i onto the diagonal of $\mathbf{H}$, equation (9) can be expressed in matrix form as follows:

$$\mathbf{H}\underline{v} - \mathbf{G}\underline{q} = -\mathbf{I}\underline{\phi}, \tag{11}$$

where $\mathbf{H}$ and $\mathbf{G}$ are the $N \times N$ matrices of the coefficients H_{ik} and G_{ik}, $\underline{v}$ and $\underline{q}$ are two vectors of order N containing the boundary values of v and $\partial v/\partial n$, respectively, $\mathbf{I}$ is an $N \times N_s$ matrix of the coefficients I_{il} and $\underline{\phi}$ is a vector of order N_s containing the values of strengths of the sources. It should be noted that a similar BEM formulation has been employed by LeNiliot and Lefèvre (2001) when applied to the heat diffusion equation.

In the two-dimensional case, and when all the boundary values for v and q are specified ($S_1 = S_2 = \Gamma$), then the system of equation (11) contains N equations and $3N_s$ unknowns, namely x_l, y_l and ϕ_l for $l = \overline{1,N_s}$. When some of the boundary values for v and q are not known ($S_1 \neq S_2$), then those values will act as unknowns in the system of equation (11), making the number of unknowns greater than $3N_s$.

It is observed that only the strengths of the sources appear linearly in the system of equation (11), while the locations of the sources appear as non-linear unknowns. In the following, we briefly present the iterative procedure used to solve the non-linear system of equation (11):

- Choose an initial guess for the locations (x_l^0, y_l^0) and the strengths ϕ_l^0 of the sources of pollution, where $l = \overline{1,N_s}$. This initial guess should be made in such a way that some bounds on the unknown variables are satisfied, namely $(x_l^0, y_l^0) \in \Omega$ and $\phi_l^0 \geq 0$, for $l = \overline{1,N_s}$, as the location of the source should be inside the solution domain and the strength of the source a positive real number.
- Separate the boundary conditions into two categories. The first set of boundary conditions contains the minimum number of boundary conditions, which generate a well-posed direct problem when combined with the governing

Helmholtz equation (3) and the initial guesses for the unknown variables made in the first step. This set of boundary conditions is used to form a constant vector $\underline{v}^A$. The second set of boundary conditions contains the information that is not necessary for solving the direct problem. These boundary conditions form a constant vector, denoted by $\underline{v}^B$, that is used in the stopping criterion of the iterative procedure.

- A positive real function of $3N_s$ variables, called the objective function, is defined as follows:

$$F(x_l, y_l, \phi_l) = \|\underline{v}^{(x_l,y_l,\phi_l)} - \underline{v}^B\|^2, \quad l = \overline{1, N_s}, \tag{12}$$

  where the vector $\underline{v}^{(x_l,y_l,\phi_l)}$ contains some of the numerical results for v and q obtained by solving the direct problems considered at each iteration, chosen such that the difference from the definition of the function F given in expression (12) is relevant.

- Solve the non-linear programming problem that is the minimization of the smooth function F subject to some bounds on the variables using a SQP method. This minimization problem is stated in the following form:

$$\text{Minimize} \;\; F(x_l, y_l, \phi_l) \quad \text{subject to} \quad \begin{cases} (x_l, y_l) \in \Omega \\ \phi_l \geq 0 \end{cases}, \quad l = \overline{1, N_s}. \tag{13}$$

This problem is solved using the NAG Fortran subroutine E04UCF, see NAG Fortran Library Manual, Mark 20. The numerical solution of the problem (13) is obtained iteratively by this subroutine containing both the value of the function F and the values of the variables x_l, y_l and ϕ_l for $l = \overline{1, N_s}$, where the function F reaches its minimal value. These last values also represent the numerical solution for our inverse source problem. This subroutine allows the user to change the accuracy of the numerical solution by modifying an optimality tolerance parameter. Broadly speaking, this parameter indicates the number of correct figures desired in the objective function F at the solution. For example, if the optimality tolerance parameter is 10^{-7} and E04UCF terminates successfully, the final value of the objective function F should have approximately seven correct figures. We mention here that in all the test examples considered the optimality tolerance parameter was taken to be equal with the machine precision, namely approximately 10^{-16}. For more details on the SQP method used by the NAG subroutine E04UCF (Powell, 1974; Gill *et al.*, 1981).

Similar approaches for solving a non-linear system of equations that arises from the application of the BEM to an inverse problem have been used for other problems (Mera and Lesnic, 2003), where this procedure has been employed in order to identify the geometry of the discontinuities in a conductive material with anisotropic conductivities.

Numerical results and discussion

In this section, we investigate several examples in order to test the method proposed in the previous section for solving the inverse source problem associated with the steady-state convection-diffusion equation. The solution domain for all the examples presented herein is chosen to be the rectangular domain $\Omega = \{(x,y) : -2 < x < 2, -1 < y < 1\}$ bounded by the rectangle $\Gamma = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3 \cup \Gamma_4$, where $\Gamma_1 = \{2\} \times (-1,1)$, $\Gamma_2 = ((-2,2) \times \{1\})$, $\Gamma_3 = \{-2\} \times (-1,1)$ and $\Gamma_4 = ((-2,2) \times \{-1\})$. This geometry is intended to approximate a region of a polluted river from one or more pollutant point sources. We consider several examples whose analytical solution is known, such that a comparison between the numerical results obtained by the method and some exact solutions can be made. In the two-dimensional case, when the boundary conditions (5) and (6) are imposed, the following analytical solutions for equations (1) and (3), respectively, are used:

$$c(x,y) = \sum_{l=1}^{N_s} \frac{\phi_l}{2\pi} \exp\left(\frac{u_1(x - x_l) + u_2(y - y_l)}{2}\right) K_0(\beta r_l) \tag{14}$$

and

$$v(x,y) = \sum_{l=1}^{N_s} \frac{\phi_l}{2\pi} \exp\left(-\frac{u_1 x_l + u_2 y_l}{2}\right) K_0(\beta r_l), \tag{15}$$

where β is given by expression (4), $r_l = \sqrt{(x - x_l)^2 + (y - y_l)^2}$ and K_0 is the modified Bessel function of order zero (Abramowitz and Stegun, 1972).

The method assumes that the number of sources, N_s, is estimated a priori and therefore, we divide the numerical results presented in this section into three different subsections, in terms of how accurate is this estimation, namely when the number of sources is estimated correctly, overestimated, or underestimated.

Some limitations of the method which are related with the value of the Péclet number of the flow under investigation are presented and explained, and finally some further analysis of the numerical results are presented.

Number of sources correctly estimated

Example 1

The first example investigated is intended to model the case of a section of river contaminated by a single source of pollution. Mathematically, the process of polluting the water is assumed to be governed by the following steady-state convection-diffusion equation:

$$\nabla^2 c(x,y) - 2\frac{\partial c}{\partial x}(x,y) + \frac{1}{2}\frac{\partial c}{\partial y}(x,y) - \frac{1}{2}c(x,y) + 2\delta((x,y),(-0.5,0.5)) = 0, \tag{16}$$

where $(x,y) \in \Omega$. The flow velocity has a dominant x component and the source is located at the point $(-0.5, 0.5)$ with a strength of 2.

Using the change of variable:

$$c = v \exp\left(\frac{4x - y}{4}\right), \tag{17}$$

equation (16) can be recast as follows:

$$\nabla^2 v(x,y) - \frac{25}{16} v(x,y) = -2\exp\left(-\frac{4x-y}{4}\right)\delta((x,y),(-0.5,0.5)). \tag{18}$$

We assume that it is known that there is only one source of pollution and that, either from measurements or by making use of some physical results, c is specified on $S_1 = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3$ and $\partial c/\partial n$ is specified on $S_2 = \Gamma_2 \cup \Gamma_4$. We observe that $S_1 \cup S_2 = \Gamma$ and $S_1 \cap S_2 = \Gamma_2 \neq \emptyset$. In practical problems, the flux $\partial c/\partial n$ on the banks of the river is equal to 0 and therefore, it is natural to consider the values of q on S_2 as known. In this example, we also assume that the concentration can be measured at some points upstream (on Γ_3) and downstream (on Γ_1) of the source of pollution and at some points on one side of the river (on Γ_2). It should be mentioned here that if c and $\partial c/\partial n$ are specified at some points, then the values of v and q, respectively, at those points are also known by employing the change of variable (17). The values of the concentration on Γ_1 and Γ_3 and the values of the flux on Γ_2 and Γ_4 are all required to form and to solve a direct problem, while the values of the concentration on Γ_2 represent the concentrations at the sampling points. In these circumstances, we wish to find the location and the strength of the source of pollution.

Having specified the boundary conditions for the inverse source problem, the analytical solutions for equations (16) and (18) are given by:

$$c(x,y) = \frac{1}{\pi}\exp\left(\frac{4(x+0.5)-(y-0.5)}{4}\right)K_0\left(\frac{5}{4}\sqrt{(x+0.5)^2+(y-0.5)^2}\right) \tag{19}$$

and

$$v(x,y) = \frac{1}{\pi}\exp\left(\frac{2.5}{4}\right)K_0\left(\frac{5}{4}\sqrt{(x+0.5)^2+(y-0.5)^2}\right), \tag{20}$$

respectively. The case investigated in this example is shown by Figure 1.

First the problem is solved in the case when exact input data are used, i.e. there are no noise in the data. The domain is discretised using 60 CBEs, i.e. 10 elements on Γ_1,

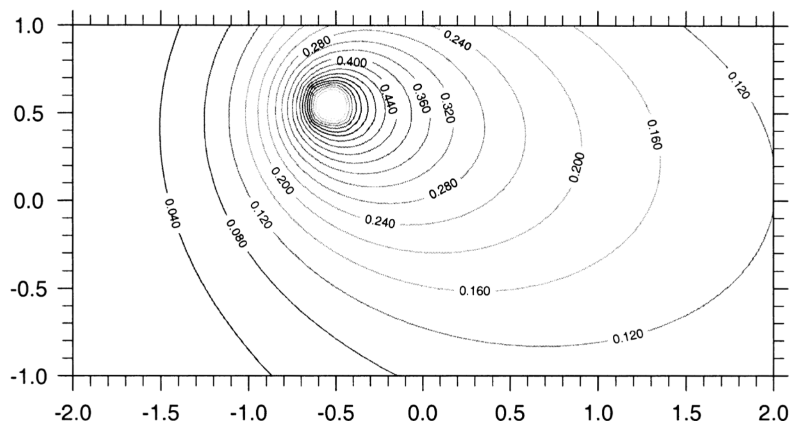

Figure 1.
Lines of constant c as given by the analytical solution for the problem considered in example 1

20 on Γ_2, 10 on Γ_3 and 20 on Γ_4. The iterative BEM proposed is now applied. The first step is to choose an initial guess for the three unknowns, for example, $x_1^0 = 0$, $y_1^0 = 0$ and $\phi_1^0 = 0$, such that $x_1^0, y_1^0 \in \Omega$ and $\phi_1^0 \geq 0$. Then the boundary conditions are separated into two categories, each category being used to define two new vectors, namely $\underline{v}^A$ containing the boundary values of v on $\Gamma_1 \cup \Gamma_3$ and the boundary values of q on $\Gamma_2 \cup \Gamma_4$, and $\underline{v}^B$ containing the boundary values of v on Γ_2. It should be noted that the vector $\underline{v}^B$ contains the observed values of v at the sampling locations, which in this case were all situated on the boundary Γ_2, i.e. the 20 boundary element nodes from Γ_2. The next step is to define the objective function F as in expression (12). The vector $\underline{v}^{(x_1,y_1,\phi_1)}$ from equation (12) contains, in this case, the numerical results for the boundary values of v on Γ_2 obtained by solving at each iteration the direct mixed problem defined by equation (18) and the following boundary conditions: v specified on $\Gamma_1 \cup \Gamma_3$ and q specified on $\Gamma_2 \cup \Gamma_4$. At the first iteration, the direct mixed problem uses the initial guesses as the values for the location and strength of the source. The NAG subroutine E04UCF is applied at this point and for each following iteration the location and strength of the source are those calculated at the previous iteration. The results obtained for the three unknowns are shown in Table I. It can be seen that even if only 60 boundary elements are employed, the numerical results are very accurate for all three unknowns and thus it can be said that the iterative procedure is accurate and convergent.

We wish now to investigate the stability of the method, and therefore the boundary values of v are perturbed by adding random noise as follows:

$$v = v + \delta v, \quad \delta v = \mathrm{G05DDF}(0, \sigma), \quad \sigma = |v| \frac{p}{100}, \tag{21}$$

where δv is a Gaussian random variable with mean zero and standard deviation σ, generated by the Fortran NAG subroutine G05DDF and p is the percentage of added noise. In a practical problem, the boundary values of q on $\Gamma_2 \cup \Gamma_4$ are not obtained from measurements, but from a physical result, namely the flux is zero on the banks of the river, and therefore they are not perturbed. For the same discretisation, namely when 60 CBEs are employed, the results are shown in Table I.

It is observed that the results remain very accurate even when noisy input data is used, such that both the location and the strength of the source of pollution are identified precisely, e.g. when 3 per cent noise is added the error in x_1, y_1 and ϕ_1 is less than 0.04, 3.47 and 3.2 per cent, respectively. For the other amounts of added noise, the error in x_1 remains very small, while the errors in y_1 and ϕ_1 are very close to the percentage of noise that was added into the input data. The fact that x_1 is found more

	Exact solution	Initial guess	0 per cent noise	1 per cent noise	3 per cent noise	5 per cent noise	10 per cent noise
x_1	-0.5	0	$-0.5 + \mathbf{O}(10^{-9})$	-0.4999	-0.4998	-0.4997	-0.4997
y_1	0.5	0	$0.5 + \mathbf{O}(10^{-9})$	0.5057	0.5173	0.5290	0.5592
ϕ_1	2	0	$2 + \mathbf{O}(10^{-8})$	1.9787	1.9363	1.8940	1.7891
$F(x_1, y_1, \phi_1)$			2.2×10^{-16}	5.9×10^{-5}	5.3×10^{-4}	1.4×10^{-3}	5.7×10^{-3}
Number of iterations			19	19	18	18	16

Table I. The numerical results obtained using 60 CBEs and input data with 0, 1, 3, 5 and 10 per cent noise, for example 1 accurately than the other two unknowns is only due to this particular example, as in other test examples the accuracy of the numerical solution was almost the same for all the three unknowns. We can also see that the smaller the noise present in the input data, the more accurate the numerical solution, and this indicates the stability of the method.

It should be mentioned here that it is not necessary that all the sampling locations be taken on the boundary of the solution domain. Several examples have been investigated for the cases when the sampling points are located inside the solution domain. In these cases $S_1 \cap S_2 = \varnothing$ and the vector $\underline{v}^B$ contains the values of the variable v at the sampling points. The iterative BEM has to be slightly modified in this case, as the objective function has to evaluate the differences between the internal values of v at the sampling points and the internal values of v obtained at each iteration.

To illustrate the type of results obtained for the case when internal sampling locations are considered, we have investigated the same pollution problem governed by equation (16). The boundary conditions consist in specifying c on Γ_1 and Γ_3, and $\partial c/\partial n$ on Γ_2 and Γ_4. Also, the values of the concentration c (and thus values of v) at the following sampling locations are considered to be known: $(-1.75, -0.75)$, $(-1.5, -0.75)$, $(-1.25, -0.75)$, $(-1, -0.75)$, $(-0.75, -0.75)$, $(-0.5, -0.75)$, $(-0.25, -0.75)$, $(0, -0.75)$, $(0.25, -0.75)$, $(0.5, -0.75)$, $(0.75, -0.75)$, $(1, -0.75)$, $(1.25, -0.75)$, $(1.5, -0.75)$ and $(1.75, -0.75)$. The numerical results obtained in this case are shown in Table II.

It can be seen that the numerical results are very similar with those shown in Table I for the case when all the sampling points were located on the boundary Γ_2. However, the values of the objective function are smaller due to the fact that in this case we had 15 sampling points, as opposed to the 20 sampling points considered when boundary-sampling locations were considered. Further, it should be remembered that the objective function is equal to the summation of the squares of the differences between the values of v at the sampling points and the values of v calculated by the direct problem at the last iteration.

Many other sampling locations were considered and in all the cases the method was able to identify an accurate and stable numerical solution. The only problem occurs when the source is very close to one of the boundaries, e.g. Γ_2, and all the sampling points are considered on the opposite boundary, e.g. Γ_4. In these cases, if the strength of the source is not large enough to send relevant information to the boundary where the sampling points are located, Γ_4, then the method is unable to provide a good numerical

Table II. The numerical results obtained using 60 CBEs and input data with 0, 1, 3, 5 and 10 per cent noise, for the internal sampling points case for example 1

	Exact solution	Initial guess	0 per cent noise	1 per cent noise	3 per cent noise	5 per cent noise	10 per cent noise
x_1	-0.5	0	$-0.5 + \mathbf{O}(10^{-9})$	-0.4999	-0.4997	-0.4997	-0.4996
y_1	0.5	0	$0.5 + \mathbf{O}(10^{-9})$	0.5057	0.5172	0.5291	0.5595
ϕ_1	2	0	$2 + \mathbf{O}(10^{-8})$	1.9787	1.9363	1.8940	1.7893
$F(x_1, y_1, \phi_1)$			1.7×10^{-16}	4.5×10^{-5}	4.0×10^{-4}	1.1×10^{-3}	4.3×10^{-3}
Number of iterations			22	22	22	22	22

solution. However, these situations can be avoided by considering sampling locations on both banks, namely Γ_2 and Γ_4.

Also it should be mentioned that when other initial guesses were considered then the same numerical results were obtained. The only difference is in the number of iterations that were needed by the iterative BEM in order to reach the numerical solution.

We wish to report that examples considering two and three sources have also been investigated and the same conclusion has been reached, namely the proposed iterative technique is convergent and stable and provides very accurate numerical solutions in cases when the a priori estimation for the number of sources is correct.

Number of sources overestimated

Example 2

This example investigates the case when three point sources are expected to be found, but the real situation corresponds to the pollution being caused by only two point sources. We wish to present the way in which the method deals with this type of situation and to compare the results with those obtained in the case when the number of sources was correctly known.

We consider that the process of pollution is governed by the following steady-state convection-diffusion equation:

$$\nabla^2 c(x,y) - \frac{\partial c}{\partial x}(x,y) - \frac{1}{2}c(x,y) + \frac{3}{2}\delta((x,y),(-1.5,0.7)) + \delta((x,y),(-1,-0.5)) = 0, \tag{22}$$

where $(x,y) \in \Omega$, which corresponds to the following equation:

$$\nabla^2 v(x,y) - \frac{3}{4}v(x,y) = -\exp\left(-\frac{x}{2}\right)\left[-\frac{3}{2}\,\delta((x,y),(-1.5,0.7)) - \delta((x,y),(-1,-0.5))\right]. \tag{23}$$

The same boundary conditions as in the previous example are prescribed, namely v is specified on $S_1 = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3$ and q is specified on $S_2 = \Gamma_2 \cup \Gamma_4$. The expressions of the analytical solutions for the partial differential equations (22) and (23) are obtained using the general expressions (14) and (15) in the following form:

$$c(x,y) = \frac{1.5}{2\pi}\exp\left(\frac{x+1.5}{2}\right)K_0\left(\frac{\sqrt{3}}{2}\sqrt{(x+1.5)^2+(y-0.7)^2}\right) + \frac{1}{2\pi}\exp\left(\frac{x+1}{2}\right)K_0\left(\frac{\sqrt{3}}{2}\sqrt{(x+1)^2+(y+0.5)^2}\right) \tag{24}$$

and

$$v(x,y) = \frac{1.5}{2\pi}\exp\left(\frac{1.5}{2}\right)K_0\left(\frac{\sqrt{3}}{2}\sqrt{(x+1.5)^2+(y-0.7)^2}\right)$$
$$+\frac{1}{2\pi}\exp\left(\frac{1}{2}\right)K_0\left(\frac{\sqrt{3}}{2}\sqrt{(x+1)^2+(y+0.5)^2}\right), \tag{25}$$

respectively. The situation investigated in this example, namely the case of pollution caused by two point sources at different locations and of different strengths, is shown by Figure 2.

On taking again 60 CBEs for the discretisation and all the unknowns to be zero as an initial guess, then the numerical results obtained when exact input data is used are presented in Table III.

We observe that the method provides very good results, by identifying the three sources as follows: the first two sources represent with an accuracy of at least $\mathbf{O}(10^{-8})$ the two real sources given by the exact solution for x_1, y_1, ϕ_1 and x_2, y_2, ϕ_2, while a third source is found at some point inside the domain, but with a very small strength which is $\mathbf{O}(10^{-10})$. It should be mentioned that the same very good accuracy has been obtained as when the number of sources was correctly estimated. However, to achieve this good accuracy in this example, namely when the number of sources is overestimated, 74 iterations were performed, resulting in a much larger computational time than that for the 37 iterations required to achieve the same very good accuracy when the method correctly knows the number of sources. Also, if the number of sources is overestimated then the matrix $\mathbf{I}$ and the vector $\underline{\phi}$ from equation (11) have a higher dimension, and therefore, their storage occupies more computational memory and their use in the mathematical operations requires more computational time.

When different initial guesses for the nine unknowns were considered, the three sources found by the method were such as one source represented very accurately the first real source given by the exact solution for x_1, y_1 and ϕ_1. The other two sources were found to be almost at the same location, with an error of $\mathbf{O}(10^{-5})$ in the location of

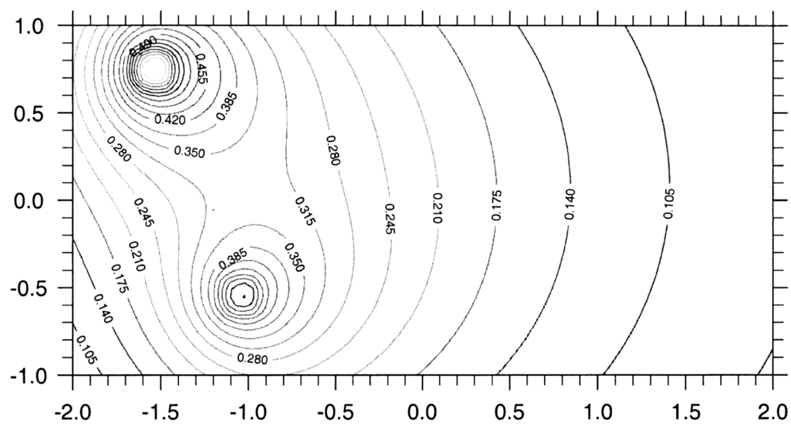

Figure 2.
Lines of constant c as given by the analytical solution for the problem considered in example 2

Table III. The numerical results obtained using 60 CBEs and input data with 0, 1, 3, 5 and 10 per cent noise, for example 2

	Exact solution	Initial guess	0 per cent noise	1 per cent noise	3 per cent noise	5 per cent noise	10 per cent noise
x_1	-1.5	0	$-1.5+\mathbf{O}(10^{-9})$	-1.4998	-1.4994	-1.4991	-1.4985
y_1	0.7	0	$0.7+\mathbf{O}(10^{-9})$	0.7020	0.7061	0.7102	0.7206
ϕ_1	1.5	0	$1.5+\mathbf{O}(10^{-8})$	1.5031	1.5090	1.5144	1.5254
x_2	-1	0	$-1+\mathbf{O}(10^{-8})$	-0.9984	-0.9956	-0.9932	-0.9886
y_2	-0.5	0	$-0.5+\mathbf{O}(10^{-8})$	-0.5101	-0.5303	-0.5505	-0.5998
ϕ_2	1	0	$1+\mathbf{O}(10^{-8})$	0.9915	0.9747	0.9583	0.9188
x_3	none	0	-1.3635	-1.5966	-1.2907	-1.2676	-1.1777
y_3	none	0	0.9142	0.3208	0.0504	0.0190	-0.2680
ϕ_3	none	0	$\mathbf{O}(10^{-10})$	$\mathbf{O}(10^{-10})$	$\mathbf{O}(10^{-10})$	$\mathbf{O}(10^{-10})$	$\mathbf{O}(10^{-11})$
$F(\underline{x}_l, \underline{\phi}_l), l=\overline{1,3}$			6.9×10^{-15}	2.0×10^{-4}	1.8×10^{-3}	5.1×10^{-3}	2.0×10^{-2}
Number of iterations			74	50	39	37	39

the second real source given by the exact values for x_2 and y_2. Also, the summation of their strengths approximated with an error of $\mathbf{O}(10^{-8})$ the second real source strength given by the exact value for ϕ_2.

Several examples have been investigated for the case when the method assumes the existence of two sources, but in reality there is only one. Two contrasting types of results have been obtained, depending on the example being considered. The first type found two sources at virtually the same location, the sum of their strengths being equal within about $\mathbf{O}(10^{-8})$ to the value of the source strength we wish to identify. The second type also found two sources, one of them being the real source, while the other source was identified at a random location inside the solution domain but with a very small strength, e.g. strength which is $\mathbf{O}(10^{-10})$. Therefore, we may conclude that the method deals very well with the situation when the number of sources is overestimated, the only inconvenience being that more computational time is required than in the case when the number of sources is known correctly.

The case when noisy input data is considered is now investigated and the results obtained are also shown in Table III. The stability of the numerical solution has again been demonstrated. We mention that only when 1 per cent noise was added into the input data was the number of iterations significantly higher than the number of iterations performed in the case when the method correctly assumed the existence of only two sources. When more noise is added into the input data then the number of iterations are almost equal, although it is still larger for the case when the method wrongly assumes the existence of three sources. However, when a different initial guess was employed, the difference between the number of iterations required to achieve a good accuracy in the two cases was more significant.

Number of sources underestimated

Example 3

This third example investigates the case of pollution caused by three different point sources, when the method wrongly assumes the existence of only two sources. Therefore, the governing convection-diffusion equation is taken to be the following:

$$\nabla^2 c(x,y) - \frac{\partial c}{\partial x}(x,y) - \frac{1}{2}c(x,y) + \frac{3}{2}\,\delta((x,y),(-1.5,0.7)) + \delta((x,y),(-1,-0.5)) + \frac{6}{5}\,\delta((x,y),(1,0.5)) = 0, \tag{26}$$

where $(x,y) \in \Omega$. The convection-diffusion equation (26) corresponds to the following equation:

$$\nabla^2 v(x,y) - \frac{3}{4}v(x,y) = -\exp\left(-\frac{x}{2}\right)\left[-\frac{3}{2}\delta((x,y),(-1.5,0.7)) - \delta((x,y),(-1,-0.5)) - \frac{6}{5}\delta((x,y),(1,0.5))\right]. \tag{27}$$

We assume that v is specified on $S_1 = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3$ and q is specified on $S_2 = \Gamma_2 \cup \Gamma_4$, which are the same type of boundary conditions used in examples 1 and 2.

With these boundary conditions the analytical solutions for the partial differential equations (26) and (27) are the following:

$$c(x,y) = \frac{1.5}{2\pi}\exp\left(\frac{x+1.5}{2}\right)K_0\left(\frac{\sqrt{3}}{2}\sqrt{(x+1.5)^2+(y-0.7)^2}\right)$$
$$+\frac{1}{2\pi}\exp\left(\frac{x+1}{2}\right)K_0\left(\frac{\sqrt{3}}{2}\sqrt{(x+1)^2+(y+0.5)^2}\right) \quad (28)$$
$$+\frac{1.2}{2\pi}\exp\left(\frac{x-1}{2}\right)K_0\left(\frac{\sqrt{3}}{2}\sqrt{(x-1)^2+(y-0.5)^2}\right)$$

and

$$v(x,y) = \frac{1.5}{2\pi}\exp\left(\frac{1.5}{2}\right)K_0\left(\frac{\sqrt{3}}{2}\sqrt{(x+1.5)^2+(y-0.7)^2}\right)$$
$$+\frac{1}{2\pi}\exp\left(\frac{1}{2}\right)K_0\left(\frac{\sqrt{3}}{2}\sqrt{(x+1)^2+(y+0.5)^2}\right) \quad (29)$$
$$+\frac{1.2}{2\pi}\exp\left(\frac{1.2}{2}\right)K_0\left(\frac{\sqrt{3}}{2}\sqrt{(x-1)^2+(y-0.5)^2}\right),$$

respectively. Figure 3 shows the case investigated in this example, namely the pollution caused by three point sources at different locations and of different strengths.

The boundary is discretised by employing 60 CBEs and the initial guesses are taken to be $x_1^0 = 0$, $y_1^0 = 0$, $\phi_1^0 = 0$, $x_2^0 = 0$, $y_2^0 = 0$, $\phi_2^0 = 0$. At each iteration, a mixed direct problem of the type presented in example 1 is solved. Exact input data is initially used and, after 38 iterations, the method generates the following numerical solution:

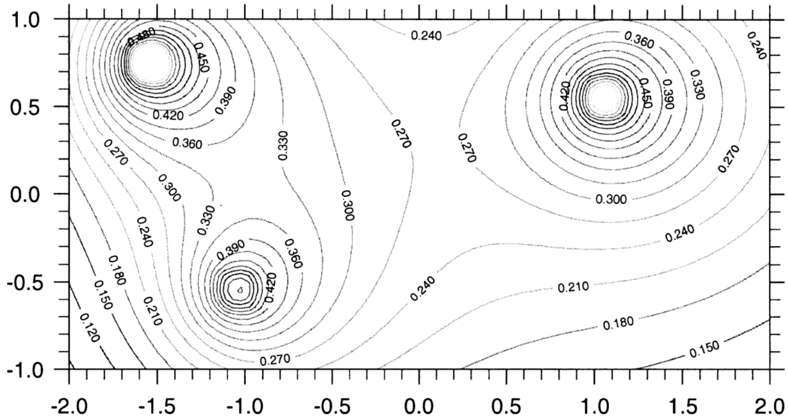

Figure 3.
Lines of constant c as given by the analytical solution for the problem considered in example 3

$x_1^0 = -0.76, y_1^0 = -0.099, \phi_1^0 = 2.27, x_2^0 = -1.55, y_2^0 = 0.75, \phi_2^0 = 1.12$, with the final objective function $F(\underline{x}_l, \underline{\phi}_l) = 4.37 \times 10^{-1}, \; l = 1, 2$. If the optimality tolerance parameter is increased the method will perform less iterations and the value of the objective function will be higher than 4.37×10^{-1}. However, we reiterate the fact that in all the examples presented in this paper, we have used the smallest possible value for the optimality tolerance parameter, namely the value of the machine precision, and therefore the numerical results cannot be further improved using this method. Other initial guesses were considered and almost the same numerical solution and final objective function were obtained, the differences being $\mathbf{O}(10^{-9})$. It is clear that this is not the solution for the problem, however, the method provides the solution for a case of pollution caused by two point sources that best fits the real situation, namely the pollution caused by three point sources. Figure 4 shows the similarity between the two cases. We have denoted with s the coordinate along the boundary of the solution domain, defined as $s = i/N$, where i is the current number of the boundary node. The numbering of the boundary elements was undertaken such that the first boundary element starts at $(2, -1)$ and then the anticlockwise direction is followed. Therefore, for example, $s = 0$ corresponds to the middle of the element starting at $(2, -1)$, and $s = 0.5$ corresponds to the middle of the element starting at $(-2, 1)$.

It is observed that although the values of the concentration for the assumed two sources case and for the given three sources case, respectively, are similar, there is still a significant difference on some parts of the boundary.

In a practical problem, when the number of sources is unknown, the final value of the objective function seems to be a very effective indicator of how relevant is the numerical solution of the method. In all the cases when the number of sources is correctly known, or when it is overestimated, the numerical solution obtained was very accurate and the objective function could be made $\mathbf{O}(10^{-15})$ or even smaller. In this case, when the number of sources is underestimated then the value of the final

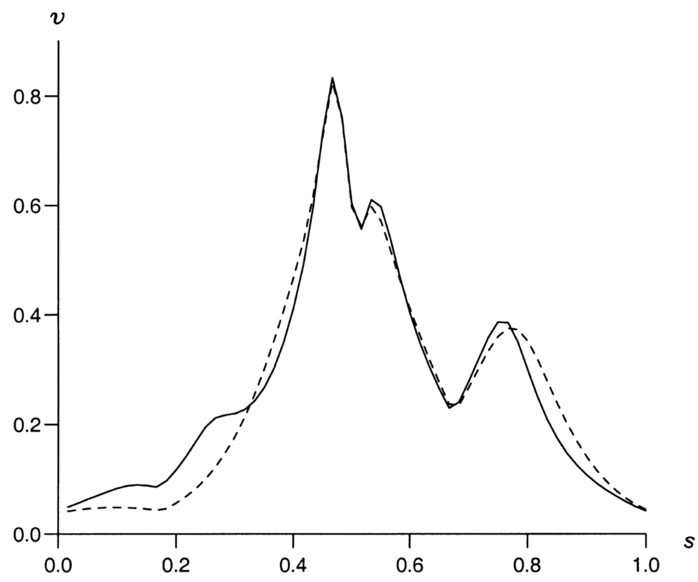

Figure 4.
The numerical results for v on Γ obtained using 60 CBEs when only two point sources are assumed (– – –), but the analytical solution (–) given by the expression (29) is for the case when there are three sources

objective function is much larger, namely 4.37×10^{-1}, which indicates that the numerical solution, although it is the best solution that can be obtained if only two sources are assumed, it is not an accurate solution for the problem.

Noisy input data has been investigated and very similar values for the location and strength of the source were obtained, as in the case when exact input data was employed. For example, when 10 per cent noise is added into the input data, the numerical solution obtained after 34 iterations is $x_1^0 = -0.78$, $y_1^0 = -0.10$, $\phi_1^0 = 2.26$, $x_2^0 = -1.53$, $y_2^0 = 0.77$ and $\phi_2^0 = 1.10$, and the final objective function is $F(\underline{x}_l, \underline{\phi}_l) = 5.02 \times 10^{-1}$, $l = 1, 2$. We recall that when the number of sources was either known, or overestimated, and data with 10 per cent noise was employed, then the final objective function was $\mathbf{O}(10^{-3})$ or $\mathbf{O}(10^{-2})$, respectively, which is at least one order of magnitude smaller than that obtained in this case. When less noise was considered then the difference between the final objective functions for the case when the number of sources was either known, or overestimated and the final objective function for the case when the number of sources was underestimated was even larger. Hence, we may conclude that even if the input data contains noise, the final value of the objective function can play the role of an indicator as to how relevant is the numerical solution.

Large Péclet number

In a practical convection-diffusion problem, the parameter that describes the relative influence of the convective and diffusive components is the Péclet number, $Pe = UL/D$, where U is a typical flow velocity, L a reference length and D the diffusivity. Throughout this paper, for simplicity, the convection-diffusion equation (1) has been considered in its non-dimensional form. For all the examples presented, the solution domain has been chosen to be the rectangular domain $\Omega = \{(x, y) : -2 < x < 2, -1 < y < 1\}$ in order to model a section of a river and the x component of the fluid flow velocity has been considered to take different small values, namely 1 and 2, making the values of the Péclet numbers 4 and 8, respectively. In these cases we have shown that the numerical method proposed in this paper is able to obtain a very accurate and stable numerical solution for the inverse source problem.

It is important to note that the same accurate and stable results have been obtained for values of the non-dimensional fluid flow velocity which were up to 25, which makes the value of the Péclet number to be ≤ 100. We have also investigated some problems with a larger flow velocity, namely 30, 40 and 50, and the only way we were able to solve such problems has been by reducing the solution domain. In this way, the method proved to be very effective. However, if the reduction of the domain cannot be achieved in such a way that it balances the large values for the fluid flow velocity (the case of $Pe > 100$), then the method presented in this paper fails. The reason for the failure of this method is the change of variable in equation (2), which transforms our non-dimensional convection-diffusion equation (1) into the Helmholtz equation with a source term equation (3). For very large values of the fluid flow velocity, this transformation concentrates all the large values of the function v to locations very close to the pollution source, while in the other parts of the solution domain the function v takes very small values. In this way, the values of v on the boundaries become smaller than the machine precision (10^{-16}) and the computer is unable to use them in mathematical operations. These boundary values represent the only input data used by the method, and the impossibility of storing their exact values causes the failure of the method.

If the iterative procedure was combined with the BEM applied directly to the convection-diffusion equation, and not to the Helmholtz equation, then it might be possible that the method would be able to solve the inverse source problem for values of the Péclet number larger than 100. Another way of overcoming this problem could be a more general approach, where the iterative procedure combines with the DRBEM in order to solve the inverse source problem associated with the steady-state convection-diffusion equation with variable coefficients.

Finally, it should be noted that if the pollution occurs in rivers with large values of the fluid flow velocity, for which the flow becomes turbulent, then the turbulent diffusivity becomes large, and hence it may limit the turbulent Péclet number to $\mathbf{O}(10^2)$.

Further analysis of the numerical results

In this section, we wish to investigate in more detail some features of the numerical solutions provided by the proposed iterative BEM in different situations.

Example 4

This example considers the same case of pollution caused by two sources that was investigated in example 2. Thus the governing convection-diffusion equation is taken to be equation (22). In this example, we assume that sampling points are considered on the boundary collocation nodes over the entire boundary Γ. As we have seen, the method can successfully deal with this type of situation not only when the number of sources N_s is correctly estimated, but also when it is over or underestimated. In all three cases the method provides numerical results for the locations and strengths of two sources. When N_s is taken to be at least 2, i.e. N_s is correctly or overestimated, then the numerical solution was found to be very accurate and stable. When N_s is taken to be 1, then the method provides a numerical solution that best bits the input data.

After a numerical solution is obtained, regardless of the quality of the estimation of N_s, a direct problem can be solved to obtained v on the boundary Γ. The values of v at the boundary collocation points obtained by solving direct problems in this way can then be compared with the values of v at the sampling points which are used as input data.

Figures 5 and 6 show the numerical values of v on the boundary Γ obtained by solving direct problems after the inverse problems suggested the locations and strengths of the pollution sources for cases when $N_s = 1, 2, 3$ and 4. These values of v are compared with the exact values of v (before the addition of the noise) that are used as input data by the inverse problems. Figure 5 is for the case when 5 per cent noise was added into the exact input data, while Figure 6 shows the case when 10 per cent noise was added into the exact input data.

As expected, it can be seen that if N_s is correctly or overestimated, then the numerical v calculated by solving direct problems after the identification of the two sources of pollution are very close to the exact values of v. On the other hand, when N_s is underestimated, the calculated and exact values of v are very different, especially on some parts of the boundary.

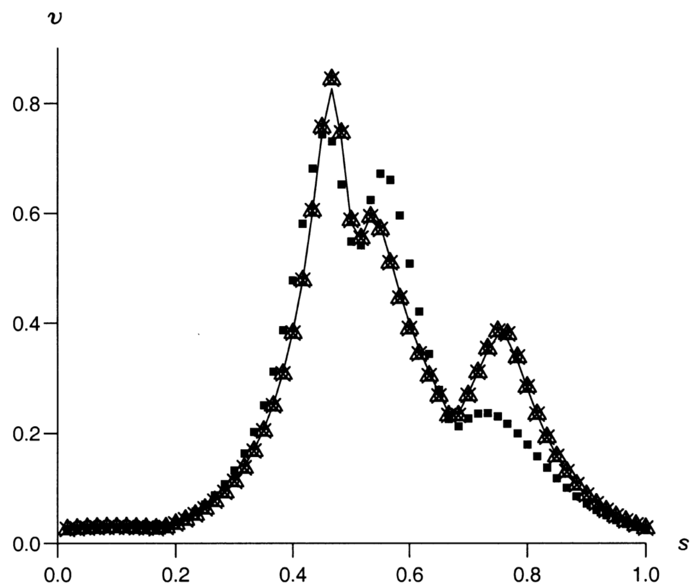

Figure 5. The numerical results for v on Γ obtained using 60 CBEs when 5 per cent noise is added into the input data for $N_s = 1(■)$, $N_s = 2(○)$, $N_s = 3(\Delta)$, $N_s = 4(\times)$ and the exact values for v used as input data (–) in example 4

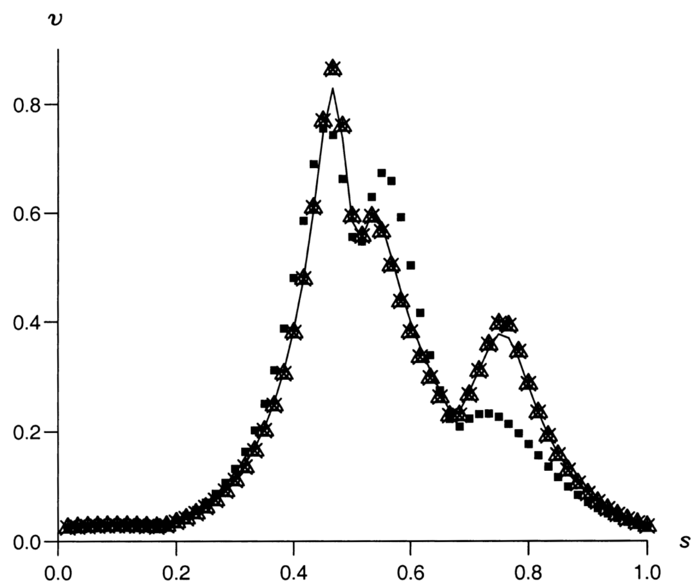

Figure 6. The numerical results for v on Γ obtained using 60 CBEs when 10 per cent noise is added into the input data for $N_s = 1(■)$, $N_s = 2(○)$, $N_s = 3(\Delta)$, $N_s = 4(\times)$ and the exact values for v used as input data (–) in example 4

The same phenomenon is observed by analysing the final objective functions and the standard deviations $\sigma(v)$ of the calculated values of v in the four cases, namely when $N_s = 1, 2, 3$ and 4 (Table IV).

We observe that for large amounts of input noise, e.g. 10 per cent, the objective function is slightly smaller when N_s is overestimated than the value of the objective function when N_s is correctly estimated. This is due to the fact that an extra dummy source with a very small strength can adjust the errors that are present in the noisy input data. However, as we have seen in the case investigated in example 2, the real situation is clearly indicated by the numerical solution and there is no danger of confusing the dummy source with a real source. We also mention that the standard deviations of the input values of v when 5 and 10 per cent noise are added into the exact input data are 4.97 and 9.95×10^{-3}, respectively. Therefore, as expected, the standard deviations of the calculated values of v in the cases when N_s is correctly or overestimated (and when a relevant numerical solution is obtained) are of the same order of magnitude with the noise, while for the underestimated case (when a non-relevant numerical solution is obtained), the standard deviation is higher.

Example 5

In this example, we wish to investigate if there are any correlations between the estimated parameters. Therefore, two cases of pollution caused by two sources are considered. The first one (denoted by 5a) is taken to be governed by the equation (22) from example 2, while the other one (denoted by 5b) is considered to be governed by the following convection-diffusion equation:

$$\nabla^2 c(x,y) - \frac{\partial c}{\partial x}(x,y) - \frac{1}{2}c(x,y) + 2\delta((x,y),(-1.5,-0.5)) + \frac{3}{2}\delta((x,y),(1,0.6)) = 0, \tag{30}$$

where $(x,y) \in \Omega$.

The same type of boundary conditions and sampling points as those from example 4 are considered. The iterative BEM is employed 10^4 times to obtain numerical solutions for each of the two cases considered. Each time, the input values of v are perturbed by the addition of 10 per cent random noise. In this way two large matrices of solutions are obtained, one for each case. Both matrices have six columns representing the six parameters that need to be identified in a two-source pollution case, namely $x_1, y_1, \phi_1, x_2, y_2$ and ϕ_2, and 10^4 rows, one for each identification problem.

We wish now to investigate some possible correlations between couples of estimated parameters in both cases. Table V shows the values of the correlation coefficients for all the couples of estimated parameters in the cases 5a and 5b.

It can be seen that although some parameters seem to be highly correlated in case 5a, e.g. ϕ_1 and ϕ_2, this high correlation is not found in case 5b. In fact this can also be observed by plotting the values of ϕ_2 as a function of ϕ_1 for all the 10^4 estimations in both cases (Figure 7). Many examples have been investigated and in some of them

Table IV. The objective function and the standard deviation for the calculated values of v for the cases considered in example 4

	5 per cent noise		10 per cent noise	
	$F(\underline{x}_l, \underline{\phi}_l), l=\overline{1,2}$	$\sigma(v)$	$F(\underline{x}_l, \underline{\phi}_l), l=\overline{1,2}$	$\sigma(v)$
$N_s = 1$	0.87	6.19×10^{-2}	1.38	6.34×10^{-2}
$N_s = 2$	5.11×10^{-3}	4.85×10^{-3}	1.25×10^{-1}	9.75×10^{-3}
$N_s = 3$	5.11×10^{-3}	4.85×10^{-3}	1.23×10^{-1}	9.75×10^{-3}
$N_s = 4$	5.11×10^{-3}	4.85×10^{-3}	1.23×10^{-1}	9.75×10^{-3}

	y_1	ϕ_1	x_2	y_2	ϕ_2
Case 5a					
x_1	−0.42	0.75	−0.07	−0.57	−0.68
y_1		0.81	−0.44	0.65	0.78
ϕ_1			0.30	−0.86	−0.97
x_2				−0.30	−0.22
y_2					0.91
Case 5b					
x_1	0.50	0.11	−0.63	−0.33	−0.14
y_1		−0.13	−0.43	−0.29	−0.01
ϕ_1			0.56	0.57	−0.73
x_2				0.71	−0.35
y_2					−0.67

Table V. The correlation coefficients for all the couples of estimated parameters in example 5

certain estimated parameters were found to be highly correlated. However, no general rule of correlation was observed, the conclusion being that these correlations are a random feature of the source identification problems.

Example 6

In this final example, we concentrate on the case of one-source pollution case governed by the following equation:

$$\nabla^2 c(x,y) - \frac{\partial c}{\partial x}(x,y) - \frac{1}{2}c(x,y) + \delta((x,y),(-1,-0.5)) = 0, \tag{31}$$

where $(x,y) \in \Omega$. We assume that in this case N_s is correctly estimated, namely it is a priori known that there is a single source of pollution. The boundary conditions and sampling points are considered the same as those from example 1. We also mention that the input data is perturbed by the addition of 10 per cent random noise.

As it was showed in example 1, the method deals very well with situations of this type, a very accurate and stable numerical solution being obtained. We wish now to focus our investigations on how sensitive is the objective function to changes in the location of the pollution source. Therefore, 3,081 points uniformly distributed inside the domain Ω were considered and, associated with these, 3,081 different inverse problems were solved. Each of these inverse problems considers that the source of pollution is enforced to be located at each of the 3,081 interior points, respectively. The same boundary conditions and sampling points were maintained for all inverse problems. Thus, each of these inverse problems is only concerned with finding the intensity of the source situated at a particular location that would best fit the values of the concentration at the sampling points. Of course, we expect to obtain the real source strength and the smallest objective function in the case when the location of the source is taken to be exactly the correct one. However, it is interesting to see what values the objective function takes when the source of pollution is enforced to be at a wrong location.

Figure 8(a) and (b) shows the contour plots of the objective functions and the source strengths, respectively, obtained by solving the above mentioned 3,081 inverse problems. It should be noted that the minimal value of the objective function is 0.005,

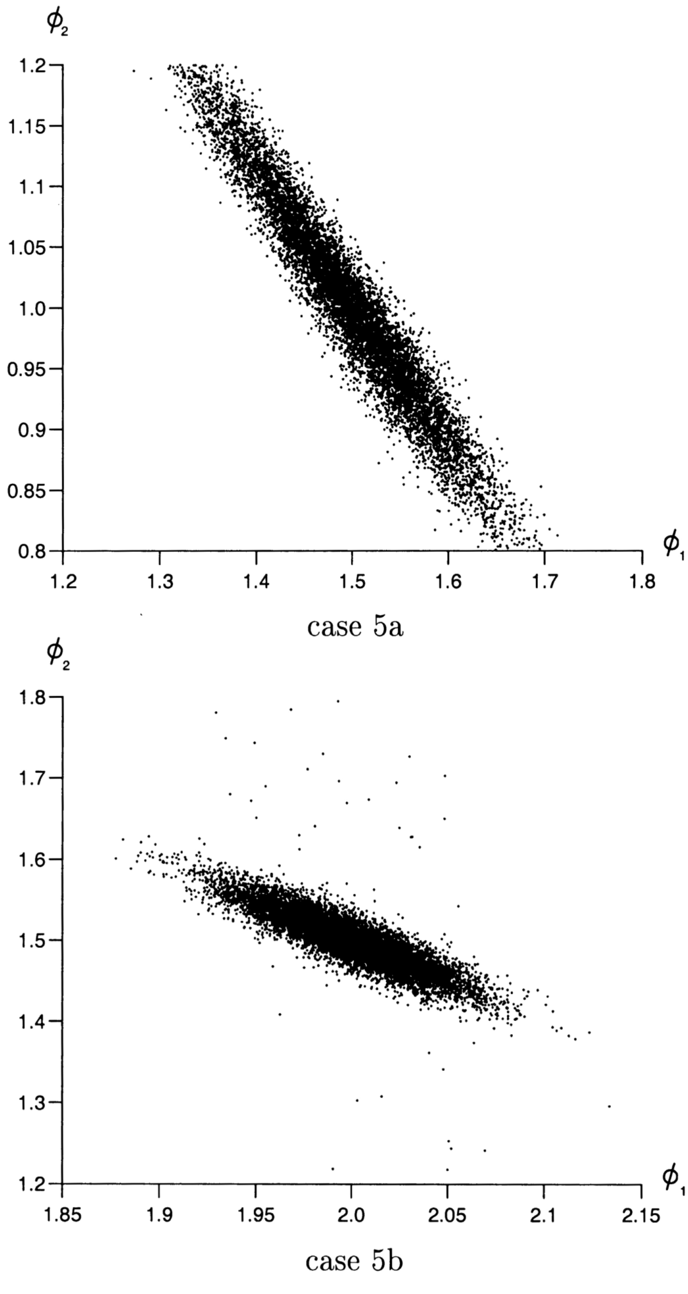

Figure 7.
Plot of ϕ_2 as a function of ϕ_1 for cases 5a and 5b, for example, 5

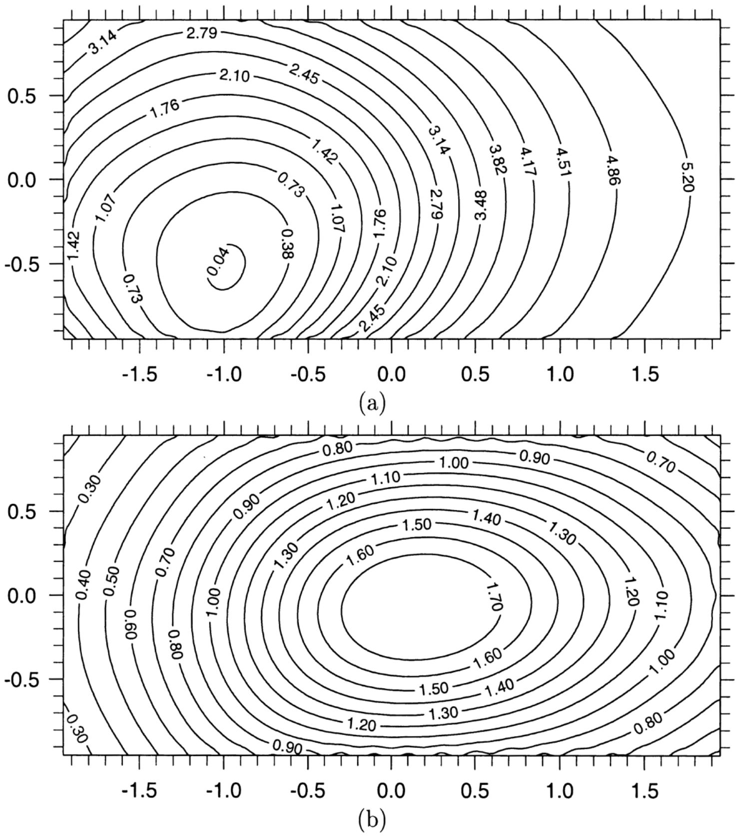

Figure 8.
Contour plots of (a) the objective function, and (b) the source strengths, for different enforced source locations inside Ω when the real source is located at $(-1, -0.5)$ and has strength 1, in example 6

and it is obtained when the source is enforced to be situated at its real location, namely at the point $(-1, -0.5)$. It can be seen that although a solution is found for each of the 3,081 inverse problems, the objective functions significantly increase when the source is enforced to be located far from the real location. From Figure 8(b) it can be observed that at the real source location the strength that best fits the input values of the concentration at the sampling points is equal, within an accuracy that was discussed in example 1, with the real source strength, namely 1. Also we can see that there are other points for which the best fitting strength is found to be 1. However, for these other points, the corresponding objective function is much larger, as it can be seen on Figure 8(a).

If we enlarge the region from Figure 8(a) where the objective function takes the smallest values, i.e. close to the real source location, it can be seen how sensitive the objective function is to even small changes in the enforced source location. This means that if a particular inverse problem is solved by minimizing the corresponding objective function, then the numerical solution for the location of the source will be in a small vicinity of the real location (Figure 9).

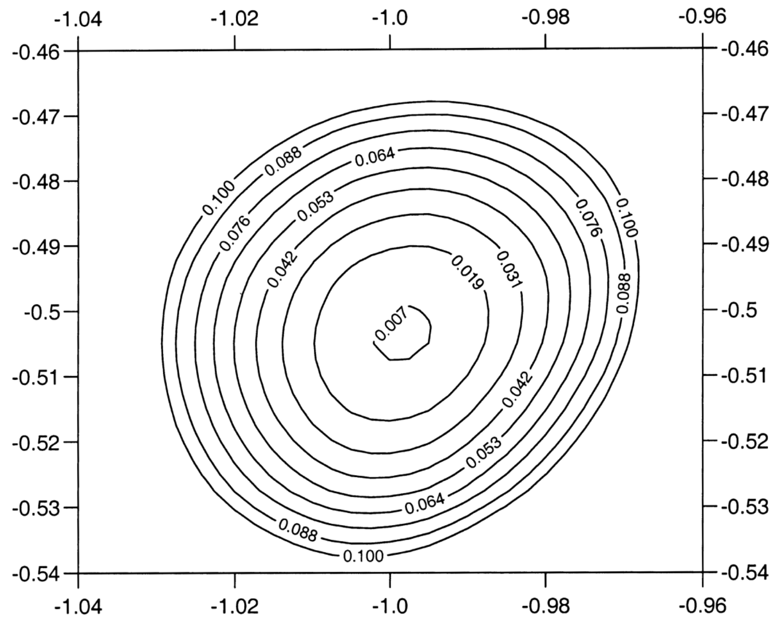

Figure 9.
Enlarged region of the contour plot for the objective function in example 6

Conclusions

In this paper, we have presented a numerical technique for solving the inverse source problem for the steady-state convection-diffusion equation with constant coefficients. However, it should be noted that although in a practical situation a water pollution problem is expected to be time-dependent, there are still many situations when in certain pollution processes the discharge of the contaminant is continuous and the system has reached the steady state. Thus the steady state convection-diffusion equation can be used to model many cases. We mention that if we consider the time-dependent convection-diffusion equation, then one would have to make some important changes in the BEM formulation considered in this paper for the steady-state case, the two problems being fundamentally different from the mathematical point of view. The method presented in this study uses a change of variable to transform the convection-diffusion equation into a Helmholtz equation with a source term. The constant BEM is then applied to this equation and the resulting non-linear system of algebraic equations is solved using an iterative procedure based on the SQP method.

The non-smooth rectangular domain has been the only geometry considered to test the computational procedure, as this type of domain is more difficult to handle than smooth domains. However, there is no reason for the method not to work for problems considered in smooth domains, such as circular or annular domains.

Some test examples have been investigated in the cases of pollution caused by one, two or three point sources in model river flows. The numerical results showed that the method provides very accurate and stable results in all these cases. We have also considered some examples when the method assumes the existence of more sources than there are in reality and again very accurate results were obtained. However, it was observed that if the number of sources is overestimated, then the method needs more computational time before achieving the very good accuracy of the numerical solution. Therefore, if possible, it is not recommended to use the method with the assumption of more sources than it is relevant for each particular practical problem under investigation. The case when the number of sources is underestimated was also investigated and it was seen that the method provides a numerical solution that best fits the real case. The final value of the objective function was observed to be an effective indicator about the relevance of the numerical solution. As a general conclusion, we may say that if the number of sources is unknown, the method can be applied for different number of sources, and the correct numerical solution taken to be that when the final value of the objective function is the smallest.

Solving the inverse source problem requires the knowledge of more information on the boundary of the domain than would be necessary for solving a well-posed direct problem. This over specified boundary data plays a very important role in the iterative procedure we have employed to solve the non-linear system of algebraic equations arised from the application of the BEM, as it is used in the stopping criterion. We have tested the method on examples where the conditions on the entire boundary, or only a part of it, were over specified. The method dealt very well with both situations and very little difference was observed between the two cases in terms of the accuracy of the results or the rate of convergence. However, we reiterate the fact that if the input values for the concentration are only available on the boundary (the possibility of dealing with these situations being an important BEM advantage), then the conditions on at least one part of the boundary have to be over specified in order for the method to work. Alternatively, instead of an over specified boundary, one may use internal sampling points.

Some limitations of the method proposed in this paper, which are related with the magnitude of the Péclet number have also been presented. It has been concluded that the method can only deal with situations when Pe is $\mathbf{O}(10^1)$. An explanation for this limitation has also been given and some possible techniques to overcome it have been suggested.

The final subsection of numerical results investigated some features of the numerical solutions, such as the difference between the calculated and the measurement values of the concentration at the sampling points, the correlations between couples of the estimated parameters and the sensitivity of the objective function to changes in the enforced source location.

Overall, we may conclude that the iterative BEM presented in this paper is well-suited for solving inverse source problems for steady-state convection-diffusion equations with constant coefficients which arise from many water pollution source identification problems.

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